

Steady transport equation in Sobolev-Slobodetskii spaces

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Abstract

We show existence of a regular solution in Sobolev-Slobodetskii spaces to stationary transport equation with inflow boundary condition in a bounded domain $\Omega \subset \mathbb{R}^2$. Our result is subject to quite general constraint on the shape of the boundary around the points where the characteristics become tangent to the boundary which applies in particular to piecewise analytical domains. Our result gives a new insight on the issue of boundary singularity for the inflow problem for stationary transport equation, solution of which is crucial for investigation of stationary compressible Navier-Stokes equations with inflow/outflow.

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1 Introduction

In this paper we are concerned with the following steady transport equation with inflow boundary condition:

$$\begin{aligned} \sigma + U \cdot \nabla \sigma &= H & \text{in } \Omega, \\ \sigma &= \sigma_{in} & \text{on } \Gamma_{in}. \end{aligned} \quad (1.1)$$

Here, Ω is a bounded domain in \mathbb{R}^2 , σ is the unknown function and U is a given vector field with normal component nonvanishing at some parts of the boundary $\Gamma = \partial\Omega$. Let us denote

$$U \cdot n(x) = d(x), \quad (1.2)$$

where n is the outward normal to Γ . Then Γ is divided into inflow part Γ_{in} , outflow part Γ_{out} and remaining part Γ_0 defined as

$$\begin{aligned} \Gamma_{in} &= \{x \in \Gamma : d(x) < 0\}, \\ \Gamma_{out} &= \{x \in \Gamma : d(x) > 0\}, \\ \Gamma_0 &= \{x \in \Gamma : d(x) = 0\}. \end{aligned} \quad (1.3)$$

Supposing that $H \in W_p^s(\Omega)$ and $\sigma_{in} \in W_p^s(\Gamma_{in})$, where W_p^s is fractional order Sobolev space equipped with Slobodetskii norm (we recall the definitions below), we show existence of a solution $\sigma \in W_p^s(\Omega)$ of the system (1.1) under certain assumption on Ω , which turns out to be quite general and apply to wide class of domains. Before we formulate the assumptions and our main result more precisely let us recall some known results concerning steady transport equations of the type of (1.1).

One of natural and important applications of equation (1.1)₁ is analysis of stationary compressible Navier-Stokes system. Namely, using the concept of effective viscous flux we can eliminate divergence of the velocity from the continuity equation which is thus reduced to steady transport equation of a type (1.1)₁. This approach was applied first in [4] in the context of regular solutions in bounded domains and later applied widely in the 90's, let us mention in particular [11] for an important improvement enabling to treat a problem exterior domain. The equation (1.1)₁ itself has also been a subject of research. Let us mention here results concerning regular solutions for small data in bounded [3] and exterior ([9], [10]) domains. A stronger result on regularity in Sobolev spaces has been shown in [6]. Weak solutions for $f \in L_p, 1 \leq p \leq \infty$, $f \in \text{BMO}$ and $f \in \mathcal{H}^1$ are studied in [2]. We shall underline that all above results concern the case of normal component of the velocity vanishing on the boundary. Admission of flow across the boundary, which is necessary if we want to apply above described approach to stationary compressible Navier-Stokes system with inflow/outflow conditions, results in substantial mathematical difficulties in the investigation of equation (1.1)₁. Namely, as the equation is solved along characteristics determined by the velocity field we have to prescribe the density on the inflow part, that is we obtain the boundary condition (1.1)₂. The above mentioned difficulties are related to the singularity which arises around the points where the characteristics become tangent to the boundary. We will denote a set of these points, called briefly singularity points, by Γ_s . With the definition (1.3) we have

$$\Gamma_s = \overline{\Gamma_0} \cap (\overline{\Gamma_{in}} \cup \overline{\Gamma_{out}}). \quad (1.4)$$

The influence of boundary singularity can be seen in [7] where the system (1.1) is applied in context of strong solutions to the steady compressible Navier-Stokes equations with inflow boundary conditions. The authors obtain a solution $u \in W_p^2, \rho \in W_p^1$ where u is the velocity of the fluid and ρ is the density. The result is subject to a constraint on the boundary around the singularity points which means roughly that the boundary must behave like a polynomial of order not higher than 2, or, in other words, that the curvature of the boundary in the singularity points must be strictly positive, which means that the characteristics cannot enter the domain too flatly. There is also a limitation on the integrability of the derivatives of the solution $2 < p < 3$. It is remarkable that similar constraints on the boundary and integrability are obtained in [13]. These constraints motivates research towards better understanding of the singularity in the system (1.1). In [12] analogous problem with inhomogeneous slip boundary condition on the velocity is considered in a rectangle where a problem of singularity does not appear and solutions in the same class are obtained without limitation on p . The system (1.1) is solved there with a technique of elliptic regularization. In [8] the result is generalized for a cylindrical domain, this time (1.1) is solved with a Lagrange-type transformation which replaces the term $u \cdot \nabla \sigma$ with a single derivative. Finally let us mention a recent result [5] where the system (1.1) is studied in context of weak

solutions with inflow condition prescribing the normal component of σU instead of σ . Existence of a solution in L_2 is shown for right hand side in L_2 and a divergence free vector field $U \in H^1$. The condition on the domain is Lipschitz continuity without additional constraints in singularity points.

In this paper we make a step towards better understanding of boundary singularity in system (1.1) working with strong solutions. Motivated by above described limitation in W_p^1 solutions we look for solutions in slightly more general class. Namely, assuming $H \in W_p^s$ which is a fractional order Sobolev space with Slobodetskii norm defined below, we show existence of a solution $\sigma \in W_p^s$ with s sufficiently small and p large enough to ensure $sp > 2$ which yields $\sigma \in L_\infty$. Our solution is hence 'slightly worse' than a W_p^1 solution, for this price we obtain existence in a wide class of domains, in particular piecewise analytical domains fits our assumptions. In order to define our solutions we need a weak formulation of (1.1).

Definition 1. By a W_p^s -solution to (1.1) we mean a function $\rho \in W_p^s(\Omega)$ such that

$$\int_{\Omega} \sigma(\phi - U \cdot \nabla \phi - \phi \operatorname{div} U) dx = - \int_{\Gamma_{in}} d\phi \sigma_{in} dS + \int_{\Omega} H \phi dx \quad \forall \phi \in C^1(\overline{\Omega}) : \phi|_{\Gamma_{out}} = 0. \quad (1.5)$$

We make the following assumptions on the velocity field U :

$$U = [u, 0], \quad u \in W_\infty^1, \quad u \geq c > 0. \quad (1.6)$$

The assumptions (1.6) are quite strong and definitely require a comment. The point is that (1.6) leads to quite simple proof of the *a priori* estimate where we see easily why the approach in fractional order spaces is natural for our problem. Therefore, introducing this assumption can be considered as a starting point of investigation of the problem (1.1) in Sobolev-Slobodetskii spaces, which is a novelty approach. Extension of our result for more general fields U leads to serious complications and the nature of singularity becomes hidden by many technicalities. It will be a subject of our forthcoming paper.

Functional spaces. We use standard Sobolev spaces W_p^k with natural k , which consist of functions with the weak derivatives up to order k in $L_p(\Omega)$, for the definition we refer for example to [1]. However, in view of Definition 1, most important for us are Sobolev-Slobodetskii spaces W_p^s with fractional s . For the sake of completeness we recall the definition here. By $W_p^s(\Omega)$ we denote the space of functions for which the norm:

$$\|f\|_{W_p^s(\Omega)} = \|f\|_{L_p(\Omega)} + \left(\iint_{\Omega^2} \frac{|f(x) - f(y)|^p}{|x - y|^{2+sp}} dx dy \right)^{1/p} \quad (1.7)$$

is finite. It is possible and will be convenient for us to express the W_p^s - norm with a sort of fractional derivatives. To this end let us denote the cuts of Ω :

$$\Omega_1(a) = \Omega \cap \{x_1 = a\}, \quad \Omega_2(b) = \Omega \cap \{x_2 = b\}.$$

Furthermore let us define

$$\underline{x}_1^* = \inf\{\underline{x}_1(x_2) : (\underline{x}_1(x_2), x_2) \in \Gamma_{in}\}, \quad \overline{x}_1^* = \sup\{\overline{x}_1(x_2) : (\overline{x}_1(x_2), x_2) \in \Gamma_{out}\}. \quad (1.8)$$

An equivalent norm in $W_p^s(\Omega)$ is defined in the following way:

Fact 1. In W_p^s we have an equivalent norm:

$$\|f\|_{W_p^s(\Omega)}^* = \|f\|_{L_p(\Omega)} + \|f\|_{W_p^s, x_1(\Omega)} + \|f\|_{W_p^s, x_2(\Omega)}, \quad (1.9)$$

where

$$\|f\|_{W_p^s, x_1(\Omega)} = \left(\int_a^b dx_2 \int_{\Omega_2(x_2)} dx_1 \int_{\Omega_2(x_2)} \frac{|f(y_1, x_2) - f(x_1, x_2)|^p}{|x_1 - y_1|^{1+sp}} dy_1 \right)^{1/p} \quad (1.10)$$

and

$$\|f\|_{W_p^s, x_2(\Omega)} = \left(\int_{\underline{x}_1^*}^{\overline{x}_1^*} dx_1 \int_{\Omega_1(x_1)} dx_2 \int_{\Omega_1(x_1)} \frac{|f(x_1, y_2) - f(x_1, x_2)|^p}{|x_2 - y_2|^{1+sp}} dy_2 \right)^{1/p}. \quad (1.11)$$

The limits of integration in outer integrals come from the definitions (1.13) and (1.8). The proof of equivalence of norms (1.7) and (1.9) can be found in [14]. Let us recall also a particular case of imbedding theorem for fractional order spaces, which will be crucial for our estimates:

Fact 2. Let $f \in W_p^s(\Omega)$ with $sp > d$ where d is the space dimension. Then $f \in L_\infty(\Omega)$ and

$$\|f\|_{L_\infty(\Omega)} \leq C \|f\|_{W_p^s(\Omega)}. \quad (1.12)$$

The domain. General case. Our result holds for a wide class of domains where the inflow and outflow parts are defined in a natural way. A general setting is to consider inflow and outflow described as

$$\begin{aligned} \Gamma_{in} &= \{(\underline{x}_1(x_2), x_2) : x_2 \in (a, b)\}, \\ \Gamma_{out} &= \{(\overline{x}_1(x_2), x_2) : x_2 \in (a, b)\}, \end{aligned} \quad (1.13)$$

with singularity points given by

$$\Gamma_s = \bigcup_{1 \leq i \leq m} \{(\underline{x}_1(x_2), x_2), x_2 = k_{in}^i\} \cup \bigcup_{1 \leq j \leq n} \{(\overline{x}_1(x_2), x_2), x_2 = k_{out}^j\}$$

where

$$a = k_{in}^1 < \dots < k_{in}^m = b, \quad a = k_{out}^1 < \dots < k_{out}^n = b.$$

Taking into account the definition of singularity points (1.4) and assumption (1.6), around the singularity points we must have

$$\lim_{x_2 \rightarrow k_{in}^{i-}} |\underline{x}_1'(x_2)| = \lim_{x_2 \rightarrow k_{in}^{i+}} |\underline{x}_1'(x_2)| = \infty,$$

$$\lim_{x_2 \rightarrow k_{out}^{j-}} |\overline{x}_1'(x_2)| = \lim_{x_2 \rightarrow k_{out}^{j+}} |\overline{x}_1'(x_2)| = \infty.$$

Furthermore we assume

$$\begin{aligned} \lim_{x_2 \rightarrow a^+} \underline{x}_1(x_2) &= c_1, & \lim_{x_2 \rightarrow b^-} \underline{x}_1(x_2) &= c_2, \\ \lim_{x_2 \rightarrow a^+} \overline{x}_1(x_2) &= d_1, & \lim_{x_2 \rightarrow b^-} \overline{x}_1(x_2) &= d_2, \end{aligned} \quad (1.14)$$

with $c_i \leq d_i$. Then we have

$$\Gamma_0 = (c_1, d_1) \times \{a\} \cup (c_2, d_2) \times \{b\} \cup \Gamma_s. \quad (1.15)$$

The domain. Simple representative case. As we shall see, the proof relies essentially on investigation of the behaviour of the boundary near the singularity points. Provided that around each singularity point condition (1.17) below is satisfied, it is enough to consider a following simple domain with inflow and outflow parts of the boundary given by (1.13), (1.14) with $a = c_i = d_i = 0$ and

$$\begin{aligned} \lim_{x_2 \rightarrow 0^+} \underline{x_1}'(x_2) &= \lim_{x_2 \rightarrow b^-} \overline{x_1}'(x_2) = -\infty, \\ \lim_{x_2 \rightarrow 0^+} \overline{x_1}'(x_2) &= \lim_{x_2 \rightarrow b^-} \underline{x_1}'(x_2) = +\infty. \end{aligned} \quad (1.16)$$

Then we have two singularity points:

$$\Gamma_s = \Gamma_0 = \{(0, 0), (0, b)\}.$$

We assume further that these are the only singularity points, that is, there are no singularity points 'inside' Γ_{in} and Γ_{out} . Similar domain is considered in [7], hence our choice facilitates a comparison of the results and techniques with above mentioned paper. Around each singularity point x_2 is given as a function of x_1 . We assume that this function satisfies the following constraint:

$$|x_2(x_1) - x_2(y_1)| \geq C|x_1 - y_1|^r \quad \text{for some } r > 1. \quad (1.17)$$

This condition is crucial in our proof and it deserves a comment. Condition (1.17) means that the boundary around the singularity points is allowed to behave in particular like a polynomial of arbitrary degree. It is worth to compare our constraint with [7] where analogous condition is required with $r = 2$, therefore our restriction is much more general. The condition (1.17) is quite technical, however it applies for a wide class of functions, in particular it holds true if the boundary around the singularity points is an analytic function. It is quite basic result but we show it for the sake of completeness.

Lemma 1. *Assume that x_2 is an analytic function of x_1 around the singularity points. Then (1.17) holds.*

Proof. By (1.16) we have $x_2'(x_1) = 0$ at the singularity points. Therefore it is enough to show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is analytic in some $[-r, r]$, $f(0) = f'(0) = 0$ and $f \neq 0$ then

$$|f(x)| \geq C|x|^N \quad \text{for } x \in (-l, l) \quad (1.18)$$

for some $C > 0$, $N \geq 2$ and $l < r$ sufficiently small. Since $f \neq 0$ and f is analytic, we must have $f^{(n)}(0) \neq 0$ for some $n \geq 2$. Let $f^{(k)}(0)$ be the first derivative not vanishing in 0. Then we have

$$f(x) = \frac{f^{(k)}(0)}{k!} x^k + R^{k+1}(x),$$

where $|R^{k+1}(x)| \leq M|x|^{k+1}$ for $x \in (-r, r)$. Hence

$$|f(x)| \geq \left| \frac{f^{(k)}(0)}{k!} \right| |x|^k - M|x|^{k+1} = \left(\frac{f^{(k)}(0)}{k!} - M|x| \right) |x|^k. \square$$

Main result. We are now ready to formulate our main result.

Theorem 1. Assume that the boundary of Ω around the singularity points satisfy (1.17) for some $C > 0$ and $N \in \mathbb{N}$. Assume U satisfy the assumptions (1.6). Assume further that $H \in W_p^s(\Omega)$ and $\sigma_{in} \in W_p^s(\Gamma_{in})$ for s, p such that $\frac{1}{r} > s > \frac{2}{p}$ where r is the exponent from (1.17). Then there exists a solution σ to (1.1) such that

$$\|\sigma\|_{W_p^s(\Omega)} \leq C[\|H\|_{W_p^s(\Omega)} + \|\sigma_{in}\|_{W_p^s(\Gamma_{in})}]. \quad (1.19)$$

2 Proof of Theorem 1

As we are concerned with linear system, the core of the proof is in appropriate estimates. The following proposition gives *a priori* estimate in W_p^s for a solution of (1.1).

Proposition 1. Assume Ω , H , U and σ_{in} satisfy the assumptions of Theorem 1. Let σ be a sufficiently regular solution to the equation (1.1). Then (1.19) holds.

Proof. First of all, let us notice that it is enough to show (1.19) for the solution of the equation

$$\begin{aligned} \tilde{\sigma}_{x_1} &= \tilde{H} \in W_p^s(\Omega), \\ \tilde{\sigma}|_{\Gamma_{in}} &= \tilde{\sigma}_{in} \in W_p^s(\Gamma_{in}). \end{aligned}$$

Indeed, if we define

$$V(x) = \int_{\underline{x}_1(x_2)}^{x_1} \frac{1}{u(s, x_2)} ds, \quad \tilde{\sigma} = e^V \sigma,$$

then we have

$$\partial_{x_1} \tilde{\sigma} = V_{x_1} e^V \sigma + e^V \sigma_{x_1}. \quad (2.1)$$

Since $V_{x_1} = \frac{1}{u}$, multiplying (2.1) by u we get

$$u \tilde{\sigma}_{x_1} = [\sigma + u \sigma_{x_1}] e^V = H e^V,$$

so

$$\tilde{\sigma}_{x_1} = \frac{H e^V}{u} =: \tilde{H}. \quad (2.2)$$

Now due to boundedness of Ω and assumptions on u we have $\frac{e^V}{u} \in W_\infty^1$ and $0 < M_1 \leq \frac{e^V}{u} \leq M_2$ in Ω , therefore

$$C_1 \left\| \frac{e^V}{u} f \right\|_{W_p^s} \leq \|f\|_{W_p^s} \leq C_2 \left\| \frac{e^V}{u} f \right\|_{W_p^s} \quad (2.3)$$

for $f \in W_p^s(\Omega)$, which shows equivalence of estimates for (2.1) and (1.1). \square

Let us proceed with the proof of Theorem 1 for (2.1) (skipping tildes for simplicity). For convenience we use the norm $\|\cdot\|_{W_p^s}^*$ (1.9), hence it is enough to find the bounds on the norms $\|\sigma\|_{L_p}$, $\|\sigma\|_{W_p^s, x_1}$ and $\|\sigma\|_{W_p^s, x_2}$ defined in (1.10) and (1.11). The idea is to express pointwise values of σ with integrals of H along the characteristics of (1.1)₁, that is, straight lines due to assumption (1.6). In fact, to operate with pointwise values we need sufficiently smooth functions,

therefore we consider smooth approximations of σ and H and use standard density argument. We start with estimate for $\|\sigma\|_{L_p}$. We have

$$\sigma(x_1, x_2) = \sigma_{in}(\underline{x}_1(x_2), x_2) + \int_{\underline{x}_1(x_2)}^{x_1} H(t, x_2) dt,$$

therefore we directly get

$$\|\sigma\|_{L_p(\Omega)} \leq C[\|\sigma_{in}\|_{L_p(\Gamma_{in})} + \|H\|_{L_p(\Omega)}]. \quad (2.4)$$

Now we consider $\|\sigma\|_{W_p^s, x_2(\Omega)}$. For convenience denote $h := y_2 - x_2$. We can assume that $\underline{x}_1(y_2) < \underline{x}_1(x_2)$, otherwise we interchange $\underline{x}_1(y_2)$ and $\underline{x}_1(x_2)$ in the integrals. We write

$$\begin{aligned} \sigma(x_1, x_2 + h) - \sigma(x_1, x_2) &= \sigma_{in}(\underline{x}_1(x_2 + h)) - \sigma_{in}(\underline{x}_1(x_2)) + \\ &\quad \int_{\underline{x}_1(x_2+h)}^{x_1} H(t, x_2 + h) dt - \int_{\underline{x}_1(x_2)}^{x_1} H(t, x_2) dt \\ &= \sigma_{in}(\underline{x}_1(x_2+h)) - \sigma_{in}(\underline{x}_1(x_2)) + \int_{\underline{x}_1(x_2)}^{x_1} [H(t, x_2+h) - H(t, x_2)] dt + \int_{\underline{x}_1(x_2+h)}^{\underline{x}_1(x_2)} H(t, x_2+h) dt = \\ &=: I_0 + I_1 + I_2, \end{aligned}$$

Hence $|\sigma(x_1, x_2 + h) - \sigma(x_1, x_2)|^p \sim |I_0|^p + |I_1|^p + |I_2|^p$. We omit the limits of integration w.r.t. x_1 and x_2 since they do not play role in the computations. Concerning the limits of $h = y_2 - x_2$, these depends on x , but what is important is integrability around 0, therefore we assume without loss of generality $h > 0$ and integrate with respect to h from 0 to some $\delta > 0$. First of all, notice that we have $dx_2 \leq dS(x_2)$ where $dS(x_2)$ is the boundary measure at Γ_{in} . Therefore

$$\begin{aligned} \int dx_1 \int dx_2 \int_0^\delta \frac{|I_0|^p}{|h|^{1+sp}} dh &\leq \int dx_1 \iint_{\Gamma_{in}^2} \frac{|\sigma_{in}(\underline{x}_1(y_2)) - \sigma_{in}(\underline{x}_1(x_2))|^p}{|x_2 - y_2|^{1+sp}} dS(x_2) dS(y_2) \\ &\leq C(\Omega) \|\sigma_{in}\|_{W_p^s(\Gamma_{in})}^p. \end{aligned} \quad (2.5)$$

Next, by Jensen inequality we have

$$\begin{aligned} \int dx_1 \int dx_2 \int_0^\delta \frac{|I_1|^p}{|h|^{1+sp}} dh &= \int dx_1 \int dx_2 \int_0^\delta dh \frac{|\int_{\underline{x}_1(x_2)}^{x_1} [H(t, x_2 + h) - H(t, x_2)] dt|^p}{|h|^{1+sp}} \\ &\leq \int dx_1 \int dx_2 \int_0^\delta dh \frac{|x_1 - \underline{x}_1(x_2)|^{p-1} \int_{\underline{x}_1(x_2)}^{x_1} |H(t, x_2 + h) - H(t, x_2)|^p dt}{|h|^{1+sp}} \\ &\leq C(\Omega) \int dx_1 \int dx_2 \int_{\underline{x}_1(x_2)}^{x_1} \int_0^\delta \frac{|H(t, x_2 + h) - H(t, x_2)|^p}{|h|^{1+sp}} dh dt \\ &\leq C(\Omega) \int dx_1 \int dx_2 \int_{\underline{x}_1(x_2)}^{\overline{x}_1(x_2)} \int_0^\delta \frac{|H(t, x_2 + h) - H(t, x_2)|^p}{|h|^{1+sp}} dh dt. \end{aligned}$$

which yields

$$\int dx_1 \int dx_2 \int_0^\delta \frac{|I_1|^p}{|h|^{1+sp}} dh \leq C(\Omega) \|H\|_{W_p^s, x_2}^p. \quad (2.6)$$

Note that the above estimate did not involve any assumption on the boundary. The assumption (1.17) will play role in the second part of the estimate. Namely, for I_2 we have

$$\begin{aligned} \int dx_1 \int dx_2 \int_0^\delta \frac{|I_2|^p}{|h|^{1+sp}} dh &= \int dx_1 \int dx_2 \int_0^\delta \frac{\left| \int_{\underline{x}_1(x_2+h)}^{\underline{x}_1(x_2)} H(t, x_2+h)^p dt \right|}{|h|^{1+sp}} dh \\ &\leq \int dx_1 \int dx_2 \int_0^\delta dh \frac{|\underline{x}_1(x_2) - \underline{x}_1(x_2+h)|^{p-1} \int_{\underline{x}_1(x_2+h)}^{\underline{x}_1(x_2)} |H(t, x_2+h)|^p dt}{|h|^{1+sp}} \\ &\leq \int dx_1 \int dx_2 \int_0^\delta dh \frac{\|H\|_{L^\infty}^p |\underline{x}_1(x_2) - \underline{x}_1(x_2+h)|^p}{|h|^{1+sp}}. \end{aligned}$$

Now condition (1.17) implies that

$$|\underline{x}_1(x_2) - \underline{x}_1(x_2+h)| \leq C|h|^\epsilon \quad (2.7)$$

where $\epsilon = \frac{1}{r}$, therefore

$$\int dx_1 \int dx_2 \int_0^\delta \frac{|I_2|^p}{|h|^{1+sp}} dh \leq \|H\|_{L^\infty}^p \int dx_1 \int dx_2 \int_0^\delta |h|^{-1+p(\epsilon-s)} dh.$$

The last integral is finite provided that $s < \epsilon$. By the imbedding theorem we get

$$\int dx_1 \int dx_2 \int_0^\delta \frac{|I_2|^p}{|h|^{1+sp}} dh \leq C \|H\|_{W_p^s}^p, \quad (2.8)$$

provided that $s < \epsilon$ where ϵ is the exponent from (2.7). On the other hand we require $s > \frac{2}{p}$. Therefore we can take $p > 2r$ where r is from (1.17) and $\frac{1}{r} > s > \frac{2}{p}$. From (2.5), (2.6) and (2.8) we conclude

$$\|\sigma\|_{W_p^s, x_2(\Omega)} \leq C[\|H\|_{W_p^s(\Omega)} + \|\sigma_{in}\|_{W_p^s(\Gamma_{in})}]. \quad (2.9)$$

It remains to estimate the fractional norm with respect to x_1 . The estimate is more direct and does not involve any assumptions on the geometry of the boundary. We have

$$\sigma(x_1+h, x_2) - \sigma(x_1, x_2) = \int_{x_1}^{x_1+h} H(t, x_2) dt.$$

Now

$$\begin{aligned} &\int dx_1 \int dx_2 \int_0^\delta dh \frac{\left| \int_{x_1}^{x_1+h} H(t, x_2) dt \right|^p}{|h|^{1+sp}} \leq \\ &\leq \int dx_1 \int dx_2 \int_0^\delta dh \frac{|h|^{p-1} \int_{x_1}^{x_1+h} |H(t, x_2)|^p dt}{|h|^{1+sp}} \leq C \|H\|_{L^\infty}^p \int_0^\delta |h|^{-1+p(1-s)} dh. \end{aligned}$$

The last integral is finite for $s < 1$. We see that

$$\|\sigma\|_{W_p^s, x_1} \leq C\|H\|_{W_p^s} \quad (2.10)$$

for $H \in W_p^s$ where $1 > s > \frac{2}{p}$, what is clearly weaker assumption than in the previous estimate. Combining (2.4), (2.9) and (2.10) we arrive at (1.19). \square

Proof of Theorem 1. As we deal with a linear system, once we have shown the estimate (1.19) the rest follows from a standard density argument. Namely, we consider a sequence of smooth functions

$$H_\epsilon \rightarrow H \quad \text{in} \quad W_p^s(\Omega).$$

Let σ_ϵ be a solution to (1.1) with H_ϵ . In particular it satisfies the weak formulation (1.5) with H_ϵ . As σ_ϵ is bounded in W_p^s , it converges weakly in W_p^s up to a subsequence (still denoted by σ_ϵ) to some $\sigma \in W_p^s$. We have to show that all the integrals in the weak formulation converge but this is obvious due to weak convergence as each integral can be understood as a functional on W_p^s . We conclude that σ is a solution to (1.1). \square

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References

- [1] Adams, R.; Fournier, J. *Sobolev spaces*, 2nd ed., Elsevier, Amsterdam, 2003
- [2] Bae, H.-O.; Choe, H. J.; Jin, B. J.; Kim, Y. *On the stationary transport equations*, Nonlinear Analysis 68 (2008), 2837-2850
- [3] Beirao da Veiga, H. *Existence results in Sobolev spaces for a stationary transport equation*. Ricerche Mat. 36 (1987), suppl., 173-184
- [4] Beirao da Veiga, H. *An L^p -Theory for the n -Dimensional, Stationary, Compressible Navier-Stokes Equations, and the Incompressible Limit for Compressible Fluids. The Equilibrium Solutions*, Comm.Math.Phys. 109 (1987), 229-248
- [5] Bernard, J.-M. *Steady transport equation in the case where the normal component of the velocity does not vanish on the boundary*. SIAM J. Math. Anal. 44 (2012), no. 2, 993-1018.
- [6] Girault, V.; Tartar, L. *L_p and W_p^1 regularity of the solution of a steady transport equation*. C. R. Math. Acad. Sci. Paris 348 (2010), no. 15-16, 885-890
- [7] Kellogg, R. B.; Kweon, J. R. *Compressible Navier-Stokes equations in a bounded domain with inflow boundary condition*, SIAM J.Math.Anal. 28,1(1997), 94-108
- [8] Mucha, P. B.; Piasecki, T. *Compressible Perturbation of Poiseuille-type flow*, Journal de Mathematiques Pures et Appliquées (9) 102 (2014), no. 2, 338-363
- [9] Novotný, A. *About steady transport equation I: L_p approach in domains with smooth boundaries*, Comment. Math. Univ. Carolin. 37 (1) (1996) 43-89.

- [10] Novotný, A. *About steady transport equation. II. Schauder estimates in domains with smooth boundaries.* Portugal. Math. 54 (1997), no. 3, 317-333
- [11] Novotný, A.; Padula, M. *L_p -approach to steady flows of viscous compressible fluids in exterior domains.* Arch. Rational Mech. Anal. 126 (1994), no. 3, 243-297
- [12] Piasecki, T. *Steady compressible Navier-Stokes flow in a square.* J. Math. Anal. Appl. 357 (2009), no. 2, 447-467
- [13] Plotnikov, P. I.; Sokolowski, J. *Shape derivative of drag functional.* SIAM J. Control Optim. 48 (2010), no. 7, 4680-4706.
- [14] Triebel, H. *Theory of function spaces.* Monographs in Mathematics, 78. Birkhauser Verlag, Basel, 1983